Enrollment No: ______________________ **Exam Seat No:**______________________

C.U.SHAH UNIVERSITY

Winter Examination-2018

Subject Name: Theories of Ring and Field

Subject Code: 5SC03TRF1 **Branch: M.Sc. (Mathematics)**

Semester: 3 **Date: 04/12/2018** **Time: 02:30 To 05:30** **Marks: 70**

Instructions:

(1) Use of Programmable calculator and any other electronic instrument is prohibited.
(2) Instructions written on main answer book are strictly to be obeyed.
(3) Draw neat diagrams and figures (if necessary) at right places.
(4) Assume suitable data if needed.

SECTION – I

Q-1 **Attempt the Following questions** **(07)**

a. Define: Ideal. **1**

b. Define: Euclidean ring. **1**

c. Write necessary and sufficient condition for the non zero element a of Euclidean ring is a unit. **1**

d. True/False. "Every principal ideal ring is Euclidean ring" **1**

e. Give an example of division ring which is not field. **1**

f. Find all subring of Z_{12}. **1**

g. Is $Z \times Q$ integral domain? **1**

Q-2 **Attempt all questions** **(14)**

a. Prove that ring of Gaussian integer is a Euclidean ring. **6**

b. Prove that characteristic of an integral domain is either zero or prime number. **5**

c. Prove that a field has no proper ideal. **3**

OR

Q-2 **Attempt all questions** **(14)**

a. Every field is a Euclidean ring. **5**

b. If a is an element of a commutative ring R with unity then prove that the set $S = \{ra \mid r \in R\}$ is a principal ideal of Rgenerated by a. **5**

c. Prove that every Euclidean ring possesses unity. **4**

Q-3 **Attempt all questions** **(14)**

a. Prove that every Euclidean ring is a principal ideal ring. **5**

b. Let R be Euclidean ring. Let a and b be two non-zero elements in R. If b is not unit in R then prove that $d(b) < d(ab)$. **5**

c. State and prove Gauss lemma. **4**

OR

Q-3		**Attempt all questions**	**(14)**
	a.	Let R be a Euclidean ring. Let $a, b \in R$ not both of which are zero. Then prove that a and b have a greatest common divisor d which can be express in the form of $d = \lambda a + \mu b$ where $\lambda, \mu \in R$.	**5**
	b.	State and prove unique factorization theorem.	**5**
	c.	Find all units of Gaussian integer.	**4**

SECTION – II

Q-4		**Attempt the Following questions**	**(07)**
	a.	Define: Irreducible polynomial.	**1**
	b.	Write the definition of Primitive polynomial.	**1**
	c.	Define: Algebraic element over field.	**1**
	d.	Define: Simple extension.	**1**
	e.	Define: Splitting field.	**1**
	f.	State Remainder theorem for polynomial.	**2**

Q-5		**Attempt all questions**	**(14)**
	a.	State and prove division algorithm for polynomials over field.	**5**
	b.	State and prove Eisenstein's criterion of Irreducibility.	**5**
	c.	Let K be an extension field of F. Let $a \in K$ be algebraic over F. Then prove that any two minimal monic polynomials for a over F are equal.	**4**

OR

Q-5		**Attempt all questions**	**(14)**
	a.	In usual notation prove that $F[x]$ is a principal ideal ring..	**5**
	b.	If L is finite extension of K and K is finite extension of F then prove that L is finite extension of F. Also $[L: F] = [L: K][K: F]$.	**5**
	c.	Let G be a subgroup of the group of all automorphisms of a field K. Then fixed field of G is a subfield of K.	**4**

Q-6		**Attempt all questions**	**(14)**
	a.	Let K be an extension field of a field F. Then prove that $a \in K$ is algebraic over F if and only if $F(a)$ is a finite extension of F.	**6**
	b.	Prove that a polynomial of degree n over a field can have at most n roots in any extension field.	**6**
	c.	Find splitting field of $f(x) = x^3 - 2 \in Q[x]$.	**2**

OR

Q-6		**Attempt all Questions**	**(14)**
	a.	State and prove fundamental theorem on Galois theory.	**6**
	b.	Let K be field of complex number and F be a field of real numbers. Then prove that K is a normal extension of F.	**6**
	c.	Show that the polynomial $x^2 + x + 4$ is irreducible over a field of integers modulo 11.	**2**